

The first cohomology of the mapping class group with coefficients in algebraic functions on the $\mathrm{SL}_2(\mathbb{C})$ moduli space

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Abstract

Consider a compact surface of genus at least two. We prove that the first cohomology group of the mapping class group with coefficients in the space of algebraic functions on the $\mathrm{SL}_2(\mathbb{C})$ moduli space vanishes.

1 Introduction

Let Σ be a compact surface, possibly with boundary, of genus at least 2, and let $\mathcal{M} = \mathcal{M}_{\mathrm{SL}_2(\mathbb{C})}$ denote the moduli space of flat $\mathrm{SL}_2(\mathbb{C})$ connections over Σ . Since \mathcal{M} may be identified with the space of $\mathrm{SL}_2(\mathbb{C})$ representations of the fundamental group of Σ modulo conjugation, \mathcal{M} has the structure of an algebraic variety. The mapping class group Γ acts on \mathcal{M} and hence on the space $\mathcal{O} = \mathcal{O}(\mathcal{M})$ of algebraic functions on \mathcal{M} , making \mathcal{O} a module over Γ . The purpose of the present paper is to prove

Theorem 1. *The first cohomology group $H^1(\Gamma, \mathcal{O})$ vanishes.*

The proof relies crucially on the Γ -equivariant identification of \mathcal{O} with another vector space on which the action of Γ is more transparent. Based on Goldman's idea of using curves in the surface to represent functions on the moduli space, Bullock, Frohman and Kania-Bartoszyńska in [BFKB99] (see also [Sko06]) proved that \mathcal{O} is Γ -equivariantly isomorphic to the complex vector space spanned by the set of multicurves on Σ . This allows one to decompose \mathcal{O} into smaller Γ modules indexed by the mapping class group orbits of multicurves.

In [AV07], we computed the cohomology group $H^1(\Gamma, \mathcal{O}^*)$, where \mathcal{O}^* denotes the algebraic dual of \mathcal{O} . Using the set of multicurves as a basis for \mathcal{O} , there is an inclusion map $\iota: \mathcal{O} \rightarrow \mathcal{O}^*$, and this induces a map on cohomology $H^1(\Gamma, \mathcal{O}) \rightarrow H^1(\Gamma, \mathcal{O}^*)$. Using the description of the target given in [AV07], we first prove that this map is zero, and then, using the results from [Vil08], we prove that it is injective.

2 Motivation

The motivation for studying the first cohomology group of the mapping class group with coefficients in a space of functions on the moduli space came from [And06]. In that paper, the first author studied deformation quantizations, or star products, of the Poisson algebra of smooth functions on the moduli space \mathcal{M}_G of flat G -connections, where $G = \mathrm{SU}(n)$. The construction uses Toeplitz operator techniques and produces a family of star products parametrized by Teichmüller space. In [And06] the problem of turning this family into one mapping class group invariant star product was reduced to a question about the first cohomology group of the mapping class group with various twisted coefficients. Specifically, one of the results in [And06] (Proposition 6) is that, provided the cohomology group $H^1(\Gamma, C^\infty(\mathcal{M}_G))$ vanishes, one may find a Γ -invariant equivalence between any two equivalent star products. Since it is easy to see that the only Γ -invariant equivalences are the multiples of the identity, this immediately implies that within each equivalence class of star products, there is at most one Γ -invariant star product.

Theorem 1 is clearly a step towards verifying the assumption above in the case of $G = \mathrm{SU}(2)$, since the $\mathrm{SU}(2)$ -moduli space is included in the $\mathrm{SL}_2(\mathbb{C})$ moduli space.

3 Splitting the coefficient module

A *multicurve* is the isotopy class of a finite collection of pairwise disjoint, simple closed curves on Σ . Let B denote the set of multicurves on Σ , and let $\mathcal{B} = \mathcal{B}(\Sigma) = \mathbb{C}B$ denote the complex vector space spanned by B . In [Sko06] one finds a complete proof of

Theorem 2. *There exists a Γ -equivariant isomorphism $v: \mathcal{B} \rightarrow \mathcal{O}$.*

If $D = \sqcup_{i=1}^n \gamma_i$ is the disjoint union of simple closed curves γ_i , $v(D)$ is simply $(-1)^n \prod_{i=1}^n f_{\tilde{\gamma}_i}$, where $\tilde{\gamma}_i$ denotes any of the oriented versions of γ_i , and $f_{\tilde{\gamma}_i}$ is Goldman's holonomy function on the moduli space.

Theorem 2 allows us to split \mathcal{O} according to the mapping class group orbits of multicurves. More precisely, for a multicurve D , let $M_D = \mathbb{C}(\Gamma D)$ denote the complex vector space spanned by the Γ -orbit through D . Then we have a decomposition as Γ -modules

$$\mathcal{O} \cong \mathcal{B} \cong \bigoplus_D M_D \tag{1}$$

where the sum is over a set of representatives of the mapping class group orbits of multicurves. This induces a corresponding decomposition of the

cohomology

$$H^1(\Gamma, \mathcal{B}) \cong \bigoplus_D H^1(\Gamma, M_D). \quad (2)$$

Hence it suffices to show that each summand on the right-hand side of (2) vanishes in order to prove Theorem 1.

4 A larger module

It turns out to ease the computation of $H^1(\Gamma, M_D)$ if one introduces a larger module. Let \mathcal{B}^* denote the algebraic dual of \mathcal{B} . Using the set of multicurves as a basis, there is a Γ -equivariant inclusion $\mathcal{B} \rightarrow \mathcal{B}^*$. In fact, we may identify \mathcal{B}^* with the space $\text{Map}(B, \mathbb{C})$ of all formal linear combinations of multicurves. There is a decomposition of \mathcal{B}^* similar to (1) into a direct product of Γ -modules,

$$\mathcal{B}^* \cong \prod_D \hat{M}_D, \quad (3)$$

where $\hat{M}_D = \text{Map}(\Gamma D, \mathbb{C})$ denotes the set of all formal linear combinations of elements of the orbit through D , and the product is over the same set of representatives as in (1).

The Γ -equivariant inclusion $\iota: M_D \rightarrow \hat{M}_D$ induces a long exact sequence in cohomology, the first part of which is

$$\begin{aligned} 0 \longrightarrow H^0(\Gamma, M_D) \longrightarrow H^0(\Gamma, \hat{M}_D) \longrightarrow H^0(\Gamma, \hat{M}_D/M_D) \\ \longrightarrow H^1(\Gamma, M_D) \xrightarrow{\iota_*} H^1(\Gamma, \hat{M}_D). \end{aligned} \quad (4)$$

In [AV07], we computed $H^1(\Gamma, \hat{M}_D)$ for any multicurve D , and showed that for any surface there exists a multicurve such that $H^1(\Gamma, \hat{M}_D)$ is non-zero.

We need the description of $H^1(\Gamma, \hat{M}_D)$ given in [AV07], so let us recall the most important facts. Let $\Gamma_D \subseteq \Gamma$ denote the stabilizer of D in Γ (permutation of the components of D are allowed). Then the Γ -equivariant identification of the set Γ/Γ_D of left cosets with the orbit ΓD induces an isomorphism of $\hat{M}_D = \text{Map}(\Gamma D, \mathbb{C})$ with the space $\text{Hom}_{\mathbb{Z}\Gamma_D}(\mathbb{Z}\Gamma, \mathbb{C})$ of $\mathbb{Z}\Gamma_D$ -homomorphisms $\mathbb{Z}\Gamma \rightarrow \mathbb{C}$.

This Γ -module is also known as the co-induced module $\text{Coind}_{\Gamma_D}^{\Gamma} \mathbb{C}$, and Shapiro's Lemma (see [Bro82]) yields an isomorphism

$$H^1(\Gamma, \hat{M}_D) = H^1(\Gamma, \text{Coind}_{\Gamma_D}^{\Gamma} \mathbb{C}) \cong H^1(\Gamma_D, \mathbb{C}) \quad (5)$$

where \mathbb{C} is a trivial Γ_D -module. Hence $H^1(\Gamma, \hat{M}_D)$ is simply the space of homomorphisms from (the abelianization of) Γ_D to \mathbb{C} .

Explicitly, the isomorphism (5) is given as follows: An element of $H^1(\Gamma, \hat{M}_D)$ is represented by a cocycle $u: \Gamma \rightarrow \hat{M}_D$, which can also be considered as a map $u: \Gamma \times \Gamma D \rightarrow \mathbb{C}$. Restricting to the subset $\Gamma_D \times \{D\} \equiv \Gamma_D$ we obtain a map $u|: \Gamma_D \rightarrow \mathbb{C}$, which is easily seen to be a homomorphism. In other words, $u|(g)$ is given by picking out the coefficient of D in $u(g)$.

5 Dehn twists and multicurves

Before starting actual computations leading to a proof of Theorem 1, we need to record a few facts regarding Dehn twists, multicurves and the modules M_D, \hat{M}_D .

5.1 Presentations and relations

It is well-known that the mapping class group is generated by Dehn twists. In fact, several finite presentations of Γ are known, where the generators are the twists in a suitable set of simple closed curves (cf. [Waj83], [Ger01]).

For later use, we mention a few relations between Dehn twists.

Lemma 3. *Dehn twists on disjoint curves commute.*

Lemma 4. *If α and β are simple closed curves intersecting transversely in a single point, the associated Dehn twists are braided. That is, $\tau_\alpha \tau_\beta \tau_\alpha = \tau_\beta \tau_\alpha \tau_\beta$.*

Lemma 5 (Chain relation). *Let α, β and γ be simple closed curves in a two-holed torus as in Figure 1, and let δ, ε denote curves parallel to the boundary components of the torus. Then $(\tau_\alpha \tau_\beta \tau_\gamma)^4 = \tau_\delta \tau_\varepsilon$.*

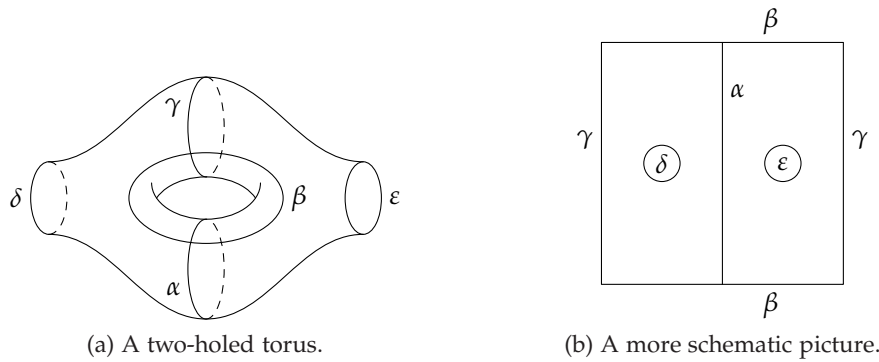


Figure 1: The chain relation.

5.2 The action of twists on multicurves

There is simple way to parametrize the set of all multicurves which was found by Dehn. For details, we refer to [PH92]. Essentially one cuts the surface into pairs of pants using $3g + r - 3$ simple closed curves γ_k , and then for each pants curve γ_k one records the geometric intersection number $m_k(D) = i(\gamma_k, D)$ (which is a non-negative integer) and a “twisting number” $t_k(D)$, which can be any integer. This defines a $6g + 2r - 6$ -tuple of integers $(m_1(D), t_1(D), \dots, m_{3g+r-3}(D), t_{3g+r-3}(D))$ (satisfying certain conditions), and, conversely, from any such tuple satisfying these conditions one may construct a multicurve.

The important fact is that in this parametrization, the action of the twist in the curve γ_k on a multicurve D is given by

$$t_k(\tau_{\gamma_k}^{\pm 1} D) = t_k(D) \pm m_k(D), \quad (6)$$

all other coordinates being unchanged. The formula (6) is intuitive in the sense that it says that for each time D intersects γ_k essentially, the action of τ_{γ_k} on D adds 1 to the twisting number of D with respect to γ_k . This can be used to prove a number of important facts.

Lemma 6. *Let γ be a simple closed curve and D a multicurve. Then the following are equivalent:*

- (1) *The twist τ_γ acts trivially on D .*
- (2) *The twist τ_γ acts trivially on each component of D .*
- (3) *The geometric intersection number between γ and D is zero.*
- (4) *One may realize γ and D disjointly.*

Conversely, if τ_γ acts non-trivially on D , all the multicurves $\tau_\gamma^n D$, $n \in \mathbb{Z}$, are distinct.

Proof. All of the above assertions can be proved from (6) by letting γ be part of a pants decomposition of the surface. This is clearly possible if γ is non-separating, while if γ is separating, observe that both connected components resulting from cutting along γ must have negative Euler characteristic (otherwise γ would be trivial or parallel to a boundary component, in which case the twist on γ clearly acts trivially on D). \square

To find a twist acting non-trivially on a multicurve, we need only find a curve which has positive geometric intersection number with the multicurve. This is possible if and only if the multicurve has a component which is not parallel to a boundary component of Σ .

5.3 Isomorphisms of modules

If D is a multicurve, let D^n denote the multicurve obtained from D by replacing each component by n parallel copies. Clearly, there are Γ -isomorphisms $M_D \rightarrow M_{D^n}$ and $\hat{M}_D \rightarrow \hat{M}_{D^n}$. Also, if γ is a simple closed curve parallel to a boundary component of Σ , we have Γ -isomorphisms $M_D \rightarrow M_{D \cup \gamma}$ and $\hat{M}_D \rightarrow \hat{M}_{D \cup \gamma}$. These observations imply that we may without loss of generality only consider multicurves without boundary parallel components, and satisfying that the multiplicities of the different components are relatively prime. Using non-standard terminology, such a multicurve will be called *reduced*.

6 The map ι_*

Let D be a reduced multicurve. The purpose of the section is to prove

Proposition 7. *The map $\iota_*: H^1(\Gamma, M_D) \rightarrow H^1(\Gamma, \hat{M}_D)$ is zero.*

The proof uses the description of $H^1(\Gamma, \hat{M}_D)$ as $\text{Hom}(\Gamma_D, \mathbb{C})$ given at the end of section 4. Let $u: \Gamma \rightarrow M_D$ be a cocycle. Since Γ is generated by Dehn twists, it is natural to study to which extent $u(\tau_\alpha)$ can contain non-zero terms on which τ_α acts trivially for simple closed curves α .

Lemma 8. *Let α be a simple closed curve on Σ , and let $E \in \Gamma D$ be a multicurve such that $\tau_\alpha E = E$. Assume that E contains at least one component which is not a parallel copy of α . Then the coefficient of E in $u(\tau_\alpha)$ is zero.*

Proof. Let ε be a component of E which is not parallel to α . Then since every component of E is disjoint from α , and since we assumed that D (and hence E) is a reduced multicurve, ε is not parallel to a boundary component of the (possibly disconnected) surface Σ_α obtained by cutting Σ along α . Hence we may find a curve β disjoint from α such that $\tau_\beta \varepsilon \neq \varepsilon$ and thus $\tau_\beta E \neq E$. Then τ_α and τ_β commute, and $u(\tau_\alpha \tau_\beta) = u(\tau_\beta \tau_\alpha)$. Using the cocycle condition this becomes

$$u(\tau_\alpha) + \tau_\alpha u(\tau_\beta) = u(\tau_\beta) + \tau_\beta u(\tau_\alpha),$$

which we may rewrite as

$$(1 - \tau_\beta) \cdot u(\tau_\alpha) = (1 - \tau_\alpha) \cdot u(\tau_\beta). \quad (7)$$

Now since $\tau_\alpha E = E$, the coefficient of E on the right-hand side of (7) is clearly 0. Assuming that $u(\tau_\alpha)$ contains some non-zero term $x E$ then implies that it must also contain the term $x \tau_\beta^{-1} E$. But since τ_α and τ_β commute, τ_α also acts trivially on $\tau_\beta^{-1} E$, so we may repeat the above argument with $\tau_\beta^{-1} E$ instead of E and conclude that $u(\tau_\alpha)$ then also contains the term $x \tau_\beta^{-2} E$.

Continuing in this way, $u(\tau_\alpha)$ contains infinitely many non-zero terms (since the multicurves $\tau_\beta^n E$ are all distinct), which is impossible since we assumed that u took values in M_D . \square

In other words, τ_α acts non-trivially on “most” of the non-zero terms occurring in $u(\tau_\alpha)$; the possible exception is when D consists of a single component and the curve α is in the orbit of D (e.g. if D and α are non-separating curves). But this possibility is easily ruled out.

Proposition 9. *Let ε be any simple closed curve. Then τ_ε acts non-trivially on any non-zero term occurring in $u(\tau_\varepsilon)$.*

Proof. By the previous lemma, we only need to prove that $u(\tau_\varepsilon)$ does not contain some non-zero term $x\varepsilon$, where ε is considered as a 1-component multicurve. To see this, observe that any curve ε can be realized as the ε occurring in the chain relation (Lemma 5); that is, there exists a genus 1 subsurface of Σ with two boundary components, one of which is ε : If ε is separating, one of the connected components obtained by cutting along ε has genus ≥ 1 , and we may if necessary choose δ to be null-homotopic. If ε is non-separating, it is always possible to find a δ such that the two curves together bound a genus 1 subsurface.

Applying the cocycle u to the chain relation, we obtain

$$u((\tau_\alpha \tau_\beta \tau_\gamma)^4) = u(\tau_\delta) + \tau_\delta u(\tau_\varepsilon).$$

But the left-hand side can be expanded (via the cocycle condition) to a sum of various actions of $\tau_\alpha, \tau_\beta, \tau_\gamma$ on the values of u on these twists; since they all act trivially on ε the coefficient of ε on the left-hand side is 0 by Lemma 8. Similarly, δ acts trivially on ε , so also the coefficient of ε in $u(\tau_\delta)$ is 0, and hence the coefficient of ε in $u(\tau_\varepsilon)$ is 0. \square

Proof (Proposition 7). Let $u: \Gamma \rightarrow M_D$ be a cocycle. By the isomorphism (5) it suffices to prove the following: For any diffeomorphism $f \in \Gamma_D$ fixing the multicurve D , the coefficient of D in $u(f)$ is zero.

Since $u|_{\Gamma_D} : \Gamma_D \rightarrow \mathbb{C}$ is a homomorphism, we may consider any power of f . Choose n sufficiently large so that f^n fixes each component of D and each side of each component. Then f^n may be realized as a diffeomorphism of the surface Σ_D obtained by cutting Σ along D . This implies that f^n can be written as a product of Dehn twists in curves not intersecting D . Hence, by Proposition 9, the coefficient of D in $u(f^n)$ is zero, and so is the coefficient of D in $u(f)$. \square

7 Proof of the main theorem

Before proving Theorem 1, we need to quote the main theorem from [Vi08]. This requires a little terminology:

Let Γ be a group and X an infinite set on which Γ acts. We define a *coloring* (or C -coloring) of X to be any map $c: X \rightarrow C$ into some set C of “colors”.

- A coloring c is *invariant* if $c(gx) = c(x)$ for each $g \in \Gamma$ and $x \in X$.
- A coloring is *almost invariant* if, for each $g \in \Gamma$, the identity $c(x) = c(gx)$ fails for only finitely many $x \in X$.
- Two colorings are *equivalent* if they assign different colors to only finitely many elements of X ; this is clearly an equivalence relation on the set of C -colorings.
- A coloring is *trivial* if it is equivalent to a monochromatic (constant) coloring.

Letting Γ denote the mapping class group of a surface of genus at least 2, and X the Γ -orbit of an arbitrary multicurve, we have

Theorem 10 ([Vil08]). *There are no non-trivial almost invariant colorings of X .*

Now we have all the tools we need.

Proof (Theorem 1). By the isomorphism (1) and the splitting (2), it suffices to prove that each summand $H^1(\Gamma, M_D)$ vanishes. By Proposition 7, we need only show that the map ι_* is injective. By the exact sequence (4), this is equivalent to proving that $H^0(\Gamma, \hat{M}_D) \rightarrow H^0(\Gamma, \hat{M}_D/M_D)$ is surjective.

Now, an invariant element of \hat{M}_D/M_D is represented by an element $v \in \hat{M}_D = \text{Map}(\Gamma D, \mathbb{C})$ such that for each $g \in \Gamma$ we have $v - gv \in M_D$. Since $(v - gv)(E) = v(E) - v(g^{-1}E)$ for $E \in \Gamma D$, we see that this must be zero for all but finitely many $E \in \Gamma D$. In other words, v must be an almost invariant \mathbb{C} -coloring of ΓD in the above language, and since by Theorem 10 no non-trivial almost invariant colorings of ΓD exist, we conclude that v is almost constant, ie. all but finitely many elements of ΓD is mapped to the same complex number z . But then v represents the same element of \hat{M}_D/M_D as the constant linear combination $\sum_{E \in \Gamma D} zE$, and hence $H^0(\Gamma, \hat{M}_D) \rightarrow H^0(\Gamma, \hat{M}_D/M_D)$ is in fact surjective. \square

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